STAT 2593

Lecture 025 - Basic Properties of Confidence Intervals

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Basic Properties of Confidence Intervals

Learning Objectives

1. Understand the need for and utility of confidence intervals.
2. Understand the general process for constructing confidence intervals.
3. Construct confidence intervals for normal population means.
4. Correctly interpret a confidence interval.
5. Use the margin of error to calculate necessary sample sizes.

Statistics is about **quantifying uncertainty** . . . we are now ready to *start* doing that.

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  - We call CIs **random intervals** as they are intervals with random endpoints.

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  - We take the lower bound, ℓ, and the upper bound u, and call $[\ell, u]$ the α-**level confidence interval**.
- We can form confidence intervals for any level α, but typically choose $\alpha \in \{0.01, 0.05, 0.10\}$.

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  - In general, easier to **standardize** taking $Z = (\overline{X} - \mu)/(\sigma/\sqrt{n}) \sim N(0,1)$.

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  - We write this often as $\overline{X} \pm 1.96 \frac{\sigma}{\sqrt{n}}$.

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  - Imagine drawing a ball from a bag containing 95 red and 5 blue marbles.

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  - Sampling sizing is an important consideration for the design of experiments.
  - If σ is not known, it will typically require an estimate.

Summary

- Confidence intervals are random intervals which quantify uncertainty in point estimation.
- Confidence intervals are formed by inverting a sampling distribution for the parameter of interest.
- Confidence intervals can be interpreted based on repetitions of the underlying experiment.
- The margin of error measures the size of a confidence interval, and is useful for sample sizing.